MATH 255, Exam 3

Name ___________________________________

Instructions No textbook, homework, calculators, phones, or smart watches may be used for this exam. A two-sided 8.5x11" note sheet is acceptable. The exam is designed to take 50 minutes and must be submitted at the end of the class period. All of your solutions should be easily identifiable and supporting work must be shown. You may use any part of this packet as scratch paper, but please clearly label what work you want to be considered for grading. Ambiguous or illegible answers will not be counted as correct.

Only the highest scoring five problems will be counted towards your total score. You cannot get over 100 points.

Problem 1 _____/20

Problem 2 _____/20

Problem 3 _____/20

Problem 4 _____/20

Problem 5 _____/20

Problem 6 _____/20

Total ______/100

There are extra pages between each problem for scratch work.
Please circle your answers!

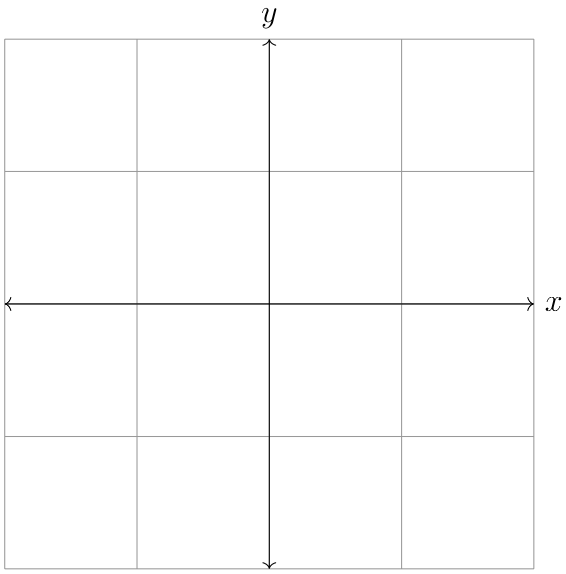
Problem 1

Consider the system of equations

$$x'(t) = x$$
$$y'(t) = y.$$

(a) Draw and clearly label the vector field for the above system at the points

- $p_1 = (0, 1, 0)$,
- $p_2 = (0, -1, 0)$,
- $p_3 = (1, 0, 0)$,
- $p_4 = (-1, 0, 0)$,
- $p_5 = (1, 1, 0)$,
- $p_6 = (1, -1, 0)$,
- $p_7 = (-1, 1, 0)$,
- $p_8 = (-1, -1, 0)$.



(b) If our initial data is $(x(0), y(0)) = (0, 0)$, how does the system evolve over time? (i.e., does the system spiral inward/outward, rotate, grow to infinity, etc.)

(c) If our initial data is $(x(0), y(0)) = (1, 1)$, how does the system evolve over time? (i.e., does the system spiral inward/outward, rotate, grow to infinity, etc.)

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Problem 2

Consider the following ODE,

$$x'(t) = x(t) \cdot t - x(t) \cdot t^2.$$

(a) Find the general solution to this differential equation.

(b) Verify that your general solution satisfies the ODE.

(c) Find the particular solution to this differential equation with the initial data $x(0) = 1$.

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Problem 3

Consider the following ODE,

$$x'(t) = (x(t) \cdot t)^2.$$

(a) Explain this ODE in words. That is, describe the rate of change the function $x(t)$ change based on the function itself and the time t.

(b) What is the order of this equation?

(c) Is this ODE linear? Why or why not?

(d) Is this ODE separable? Why or why not?

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Problem 4

Consider the following differential equation

$$x''(t) = x'(t) \cdot t + x(t) \cdot t^2.$$

(a) Is this ODE linear? Why or why not?

(b) Define a new function $y(t) = x'(t)$. Using this new function, write the ODE as a system of two first-order equations.

(c) Write this system as a matrix/vector equation

$$\mathbf{v}' = M(t)\mathbf{v}.$$

Note that the matrix $M(t)$ will depend on t.

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Problem 5

Consider the nonlinear system of equations

$$x'(t) = x(t) - x(t)y(t) + y(t)$$
$$y'(t) = (x(t)^2 - x) - (y(t)^2 - y).$$

(a) Find the matrix of the linearization about the point $(x, y) = (0, 0)$.

(b) Show that $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$ are eigenvalues of the matrix you found in (a).

(c) Based on the eigenvalues, describe the behavior of the system (i.e., does the system spiral inward/outward, rotate, grow to infinity, etc.).

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Problem 6

Consider the 2-dimensional heat equation for a function $u(x, y, t)$ given by

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0;$$

- on the region $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$;
- with initial conditions $u(x, y, 0) = \sin(x)\sin(y)$;
- with boundary conditions

$$u(0, 0, t) = u(0, \pi, t) = u(\pi, 0, t) = u(\pi, \pi, t) = 0.$$

We want to show that the function $u(x, y, t) = e^{-2t}\sin(x)\sin(y)$ solves this problem.

(a) Show that the given $u(x, y, t)$ solves the PDE.

(b) Show that the given $u(x, y, t)$ solves the initial conditions.

(c) Show that the given $u(x, y, t)$ solves the boundary conditions.

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